Spectrum condition, analyticity, Reeh-Schlieder and cluster properties in non-relativistic Galilei-invariant quantum theory

1982 J. Phys. A: Math. Gen. 15 3715

(http://iopscience.iop.org/0305-4470/15/12/023)

J. Phys. A: Math. Gen. **15** (1982) 3715–3723. Printed in Great Britain

Spectrum condition, analyticity, Reeh–Schlieder and cluster properties in non-relativistic Galilei-invariant quantum theory

Manfred Requardt
Institut für Theoretische Physik, Universität Göttingen, 34 Göttingen, West Germany

Received 16 March 1982, in final form 28 June 1982

Abstract. We show that properties like spectrum condition, analyticity of n-point functions, space-like clustering of correlation functions stronger than any inverse power, the Reeh–Schlieder property do hold in Galilei invariant quantum theory. Furthermore, the range of validity of Haag's theorem is briefly discussed. The results seem to be of relevance both in the non-relativistic regime proper and as a hint that many of the properties typically attributed to relativistic field theories are actually a common feature of every theory with a zero mean-particle density and translation-invariant Hamiltonian.

1. Introduction

In this paper we should like to show that several quite useful properties which are at the basis of general relativistic quantum field theory also hold in a large part of the non-relativistic regime. These properties are the well known spectrum condition, analyticity of n-point functions, the Reeh–Schlieder property and strong space-like clustering.

Our motivation is twofold. For one thing we want to show that, contrary, perhaps, to widespread belief, the well known Wightman axioms of relativistic quantum field theories are not characteristic for relativistic theories but are typical properties of systems having a zero particle density and a Hamiltonian commuting with translations. The usual Schrödinger theory also belongs to this class if treated properly, that is, if the interaction potential is not pinned at the origin. So the results shed some light on the fundamentality of some assumptions of field theory and their true origin.

Furthermore, we believe that there are useful applications in the non-relativistic regime itself. Beside the usual n-body quantum theory, theories are also covered where particles are created and annihilated, that is, systems which are low-energy limits of the relativistic theory like the so-called 'Galilee model' as well as fully non-relativistic theories where e.g. molecules are allowed to disintegrate and recombine in scattering processes.

In particular, the proof of the existence of strong space-like clustering of states at different times seems to be new even in the well known Schrödinger theory.

In the following we deal exclusively with theories which admit a unitary representation of the Galilei group. This class of theories allows for a rather complete characteri-

0305-4470/82/123715+09$02.00

sation and is, on the other hand, highly non-trivial from the point of physics which will be exemplified in the following section. Systems with a finite mean particle density where the Galilei group is spontaneously broken (see e.g. Swieca 1967) will be discussed elsewhere.

The paper is organised as follows. In § 2 the physical representations of the Galilei group are discussed and the role of Haag's theorem is clarified. In § 3 a general spectrum condition is proven which allows for a derivation of analyticity properties in § 4 and by which the proof of a Reeh–Schlieder property is accomplished. In § 5 we prove a strong cluster property with respect to the space coordinates of n-point functions.

2. Physical representations of the Galilei group and Haag's theorem

Contrary to the Poincaré group, the physically relevant representations in the non-relativistic regime are the representations of the central extension of the Galilei group (for a rather complete discussion see Lévy-Leblond (1971) and the references therein). The central element which serves as a superselection observable turns out to be the mass operator M. M commutes with the Hamiltonian, hence physics can be done in every mass superselection sector separately.

It is a characteristic of these representations that the quantum fields carry a phase factor as in gauge theory:

$$\mathrm{U}(g)\psi(\boldsymbol{x}, t)\mathrm{U}(g)^{-1} = \exp[\mathrm{i}m(\tfrac{1}{2}v^2 t + \boldsymbol{v}\cdot\boldsymbol{R}\boldsymbol{x})]\psi(g^{-1}(\boldsymbol{x}, t)) \tag{1}$$

where $\boldsymbol{v}$ denotes the velocity, R rotations and m the mass of the field.

In Lévy-Leblond (1967) it is shown, by giving counter examples, that Haag's theorem need not hold in the non-relativistic regime. As long as the mass superselection rule is respected, one can even construct theories which allow particle creation and annihilation processes. One reason that one can construct interacting theories even in Fock space, results from the peculiar structure of the Lie algebra of the extended Galilei group where the Hamiltonian never occurs on the right-hand side of the commutation relations. On the other side the deeper reason is not so obvious since there exists an extension of Haag's theorem to theories invariant under the Euclidean group (see e.g. Streit 1969, Emch 1972, or the review article of Wightman 1964). Studying these articles Haag's theorem seems to be inescapable but there is a somewhat hidden assumption which seems completely natural to a physicist accustomed to relativistic quantum theory. Haag's theorem holds under the assumption that the canonical momenta of the fields ψ_α are just $\partial_t\psi_\alpha$.

This assumption, on the other hand, is frequently not fulfilled in the non-relativistic regime. Hence a construction of non-trivial field theories in Fock space is possible. Theories with a non-zero particle density, especially with temperature $T \neq 0$ are more problematical. Here the correct representations of the field algebra seem to carry a 'particle' and 'hole' structure, in other words, creation and annihilation operators show up together in the field operators, thus reestablishing a kind of particle–antiparticle symmetry. This phenomenon is connected with $\pi_\alpha = \partial_t\psi_\alpha$. We do not want to dwell any more on this interesting point here since it will be discussed elsewhere. In any case the discussion shows that non-trivial quantum field theories are possible like second quantisation of quantum theory, the Galilee model (Lévy-Leblond 1967) etc.

3. Support properties of the energy–momentum spectrum

We discuss exclusively, in this paper, the case where the particle number operators for the various particle sorts N_i do exist. The Hilbert space $\mathcal{H}$ is assumed to be the norm closure of the linear hull of the states with an arbitrary but finite number of particles, furthermore a vacuum state Ω is to exist which is invariant under the representation of the extended Galilei group U(g).

Let H be the Hamilton operator, $\boldsymbol{P}$ the momentum operator. We assume throughout the paper $[H, \boldsymbol{P}] = 0$, hence somewhat sloppily $[\mathrm{d}E_\omega, \mathrm{d}E_{\boldsymbol{k}}] = 0$ where $\mathrm{d}E_\omega$, $\mathrm{d}E_{\boldsymbol{k}}$ are the spectral measures of H, $\boldsymbol{P}$. In the following we want to locate the joint spectrum of $(H, \boldsymbol{P})$ in $\mathbb{R}^4$. To this end we define the subspace $\mathcal{H}_k \subset \mathcal{H}$.

$$\mathcal{H}_k := \int_{|\boldsymbol{k}| \geqslant k} \mathrm{d}E_{\boldsymbol{k}} \mathcal{H} \tag{2}$$

$H \upharpoonright \mathcal{H}_k$ is again an SA operator on $\mathcal{H}_k$; $H_k := Q_k H Q_k$ with Q_k the projection on $\mathcal{H}_k$.

The idea is to locate the spectrum of H_k in $\mathcal{H}_k$. Let H_0 be the free Hamiltonian, that is

$$H_0 := \int \sum_i \frac{1}{2m_i} \nabla \psi_i^+ \nabla \psi_i \, \mathrm{d}\boldsymbol{x} \tag{3}$$

where the sum extends over the different types of particles occurring in the theory. The Fourier transformed version reads

$$H_0 = \int \sum_i \frac{1}{2m_i} k^2 a_i^+ a_i \, \mathrm{d}\boldsymbol{k}. \tag{4}$$

(In the following we shall deal, for simplicity, only with the simple dispersion law $\omega \sim k^2$ but a more general $\omega(k)$ would not do any harm.)

Since $\boldsymbol{P}$ commutes with the various N_i's $\mathcal{H}_k$ can again be written as a direct sum of finite particle subspaces. Let $\mathcal{H}\{n_i\}$ denote the subspace with definite numbers n_i of the various particles. For the free motion $[H_0, N_i] = 0$, that is the support of the H_0 spectrum in $\mathcal{H}_k \cap \mathcal{H}\{n_i\}$ is

$$\left\{ \omega \middle| \omega = \sum_i \sum_{j_i=1}^{n_i} \frac{1}{2m_i} k_{j_i}^2 ; \left| \sum_i \sum_{j_i} \boldsymbol{k}_{j_i} \right| \geqslant k \right\}. \tag{5}$$

With $m := \sup\{m_i\}$ we have

$$\sum_i \sum_{j_i} \frac{1}{2m_i} |\boldsymbol{k}_{j_i}|^2 \geqslant \frac{1}{2m} \sum_i \sum_{j_i} |\boldsymbol{k}_{j_i}|^2 \geqslant \frac{1}{2m} \frac{1}{N\{n_i\}} \Big(\sum_i \sum_{j_i} |\boldsymbol{k}_{j_i}| \Big)^2$$

$$\geqslant \frac{1}{2m} \frac{1}{N\{n_i\}} k^2 \qquad \text{with } N\{n_i\} := \sum_i n_i. \tag{6}$$

(In the second inequality the Cauchy–Schwartz inequality has been employed.)

Hence the $(H_0, \boldsymbol{P})$ spectrum is bounded from below in every finite particle subspace $\mathcal{H}_M$ by the hypersurface $\omega = (1/2mN)k^2$. (One can get an analogous result by separating the motion into centre-of-mass and relative coordinates.) On the other hand, while Galilei-invariant interactions H_I are allowed to change the particle numbers n_i of a state $\psi\{n_i\}$ the mass of the state $M = \sum n_i m_i$ has to be conserved, that is, the Hamilton operator leaves every subspace $\mathcal{H}_M$ with definite mass M invariant. This motivates the following weak assumption (7) and has as a consequence the simple but useful lemma 1.

With $H = H_0 + H_I$ SA and $\mathscr{H} = \Sigma_M \mathscr{H}_M$ we assume H_I to be H_0-small in every subspace $\mathscr{H}_M$ with relative bound $a_M < 1$, that is:

$$\|H_I(\psi_M)\| \leq a_M \|H_0 \psi_M\| + b_M \|\psi_M\| \qquad \psi_M \in D_{H_0} \restriction \mathscr{H}_M \subset D_{H_I} \restriction \mathscr{H}_M. \tag{7}$$

Furthermore, $H_I = H - H_0$ commutes with $\boldsymbol{P}$, hence

$$\|Q_k H_I Q_k(\psi_{M,k})\| \leq a_M \|Q_k H_0 Q_k(\psi_{M,k})\| + b_M \|\psi_{M,k}\| \qquad \psi_{M,k} \in \mathscr{H}_M \cap \mathscr{H}_k. \tag{8}$$

Lemma 1. In every sector $\mathscr{H}_M$ the number of particles is bounded from above by $M \cdot m'^{-1}$ and from below by $M \cdot m^{-1}$ with $m' := \min\{m_i\}$, $m := \max\{m_i\}$.

This enables us to prove the following theorem.

Theorem 1. With H_k, $\mathscr{H}_k \cap \mathscr{H}_M$ as defined above and under the condition (7) H is bounded from below in every $\mathscr{H}_k \cap \mathscr{H}_M$ with the bound

$$\langle H \rangle_{\mathscr{H}_k \cap \mathscr{H}_M} \geq \tilde{M}_k - \max\left\{\frac{b_M}{1 - a_M}, a_M \tilde{M}_k + b_M\right\}$$

with

$$\tilde{M}_k := \inf \langle H_0 \rangle_{\mathscr{H}_k \cap \mathscr{H}_M} \geq \frac{k^2}{2m(Mm'^{-1})} = \frac{k^2}{2M} \frac{m'}{m}.$$

Proof. This is an application of the Kato–Rellich theorem to the above situation. The proof consists roughly of an investigation for what $z \in \mathbb{C}$

$$\frac{1}{H - z} = \frac{1}{H_0 - z}\left(1 + H_I \frac{1}{H_0 - z}\right)^{-1} \tag{9}$$

is well defined as a bounded operator where the H_0 smallness is used to infer the boundedness of the second factor on the RHS (see e.g. Kato 1966, Reed and Simon 1975).

Since in our case the lower bound of H_0 in $\mathscr{H}_k$ is easily controllable we can weaken the condition (7) considerably. Let H_I be given as a symmetric quadratic form on the form domain $Q(H_0 \restriction \mathscr{H}_M)$ of $H_0 \restriction \mathscr{H}_M$ for all M with

$$(\psi | H_I \psi) \leq a_M (\psi | H_0 \psi) + b_M (\psi | \psi) \qquad a_M < 1 \qquad \psi \in D(H_0 \restriction \mathscr{H}_M) \tag{10}$$

then $H \restriction \mathscr{H}_M$ can be defined as a unique SA operator with $Q(H \restriction \mathscr{H}_M) = Q(H_0 \restriction \mathscr{H}_M)$ with the help of the (*KLMN*) theorem (see e.g. Reed and Simon 1975) and with general lower bound $-b_M$. In the special case discussed in this paper where we are interested in the reduced case $H \restriction \mathscr{H}_k \cap \mathscr{H}_M$ we are in a better position.

Theorem 2. With the definitions as in theorem 1 and (10) we have

$$\langle H \rangle_{\mathscr{H}_k \cap \mathscr{H}_M} \geq (1 - a_M)\tilde{M}_k - b_M \geq (1 - a_M)\frac{k^2}{2M}\frac{m'}{m} - b_M.$$

Proof. This follows from an inspection of the proof of the (*KLMN*) theorem in Reed and Simon (1975).

Remark. Note that the class of allowed interactions in the second theorem is much larger than in theorem 1. On the other hand, the domain properties (e.g. domains of essential self-adjointness) of the various operators are much more involved.

We shall take advantage of the two theorems in the following corollary.

Corollary 1. The joint spectrum of $(H, \boldsymbol{P})$ in every subspace with definite mass $\mathscr{H}_M$ is bounded from below by the hypersurface

$$\omega = \frac{1}{2M}\frac{m'}{m}(1-a_M)k^2 - b'_M.$$

Proof. The case of theorem 2 is obvious. As for theorem 1 we have for sufficiently large k: $a\tilde{M}_k + b \geqslant b/(1-a)$, hence with a redefined b' we have the above statement since the spectrum is bounded from below for arbitrary k.

Thus we see that the existence of a spectrum condition is not a typical relativistic phenomenon. In every subspace $\mathscr{H}_M$ we have an analogous condition in our non-relativistic theories. Since the subspaces for different M are mutually orthogonal this is almost the same as the full spectrum condition on the whole $\mathscr{H}$. But it can be seen from (6) that for $M \to \infty$, that is $N \to \infty$, the parabola which confines the spectrum from below becomes flatter and flatter. This derives from the particle number dependence of this bound which can not be removed. For $N \to \infty$ we can distribute the various momenta $\boldsymbol{k}_i$ in such a way that $|\Sigma_i \boldsymbol{k}_i| > k$ but $\Sigma_i k_i^2/2m_i \to 0$. Hence on physical grounds there can be no overall spectrum condition.

In the following section we want to employ these results to derive analyticity properties for the n-point functions of the theory. To this end we can put the results in a more appropriate form.

Corollary 2. The joint spectrum of $(H, \boldsymbol{P})$ in every $\mathscr{H}_M$ can be embedded in a domain $K_M \cup \Gamma$ where K_M is a sphere with sufficiently large diameter with centre $(0, \boldsymbol{0}) \in \mathbb{R}^4$ and Γ the forward cone $\{(\omega, \boldsymbol{k}); \omega \geqslant |\kappa|\}$.

Proof. The proof is obvious since for sufficiently large $|\boldsymbol{k}|$ the parabola under discussion intersects every forward cone with arbitrary apex angle.

4. Analyticity properties of n-point functions and a Reeh–Schlieder theorem

We shall start with the 2-point function

$$F(t, \boldsymbol{x}) := (\varphi|\mathrm{e}^{-\mathrm{i}Ht}\,\mathrm{e}^{\mathrm{i}\boldsymbol{P}\boldsymbol{x}}\psi) \qquad \varphi, \psi \in \mathscr{H}_M. \tag{11}$$

The Fourier transform $\tilde{F}(\omega, \boldsymbol{k})$ is a measure. With $F(\omega, \boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k} = (\varphi|\mathrm{d}E_H(\omega)\,\mathrm{d}E_{\boldsymbol{P}}(\boldsymbol{k})\psi)$ it is obvious that the support of $\tilde{F}$ is contained in $K_M \cup \Gamma$, hence:

$$F(t, \boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_{K_M \cup \Gamma} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\tilde{F}(\omega, \boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k} = \int_{K_M \cup \Gamma \setminus \Gamma}(\cdots)\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k} + \int_{\Gamma}(\cdots)\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k} \tag{12}$$

Since $K_M \cup \Gamma \setminus \Gamma$ is bounded, the analytic continuation with respect to $(t, \boldsymbol{x})$ into the whole $\mathbb{C}^4$ is trivial for the first expression on the RHS. To continue the second integral we require that the expression remains bounded for a certain domain $\subset\mathbb{C}^4$. With $z_0 := t - \mathrm{i}\tau$, $\boldsymbol{z} := \boldsymbol{x} - \mathrm{i}\boldsymbol{y}$ the condition $(\omega\tau - \boldsymbol{k}\boldsymbol{y}) > 0$ is sufficient with $(\omega, \boldsymbol{k}) \in \Gamma$. This entails $(\tau, \boldsymbol{y}) \in \mathring{\Gamma}$, $\mathring{\Gamma}$ the open kernel of Γ. On the other hand every cone $\{\omega > c|\boldsymbol{k}|, c > 0\}$ would be sufficient in (12) as indicated in the proof of corollary 2 provided K_M is chosen large enough. Hence F is analytic in a much greater domain, namely for $c \to \infty$ the union of dual cones $\tilde{\Gamma}_c$ which are defined as the sets $\{(\tau, \boldsymbol{y}); \omega\tau - \boldsymbol{k}\boldsymbol{y} > 0$ with $(\omega, \boldsymbol{k}) \in \Gamma_c\}$ is simply $\{(\tau, \boldsymbol{y}); \tau > 0\}$. Thus F is actually analytic in $\mathbb{R}^4 - \mathrm{i}\{(\tau, \boldsymbol{y}); \tau > 0\} =: T$.

Theorem 3. With F defined by (11) it is analytic with respect to $(z_0, \boldsymbol{z}) = (t + \mathrm{i}\tau, \boldsymbol{x} + \mathrm{i}\boldsymbol{y})$ in the domain $\mathbb{R}^4 - \mathrm{i}\{(\tau, \boldsymbol{y}); \tau > 0\}$. $F(t, \boldsymbol{x})$ are the boundary values of $F(z_0, \boldsymbol{z})$ for $\mathrm{Im}(z_0, \boldsymbol{z}) \to 0$.

Remark. The fact that one has analyticity in t when H is bounded from below was exploited in Hegerfeld *et al* (1980) to infer e.g. support properties of eigenfunctions in Schrödinger-like theories.

The generalisation to n-point functions is now immediate. We assumed that the set of $\{n_i\}$-particle states is dense in $\mathscr{H}$. Furthermore we showed that $(H, \boldsymbol{P})$ can be split into a direct sum $\Sigma_M H_M$, $\Sigma_M \boldsymbol{P}_M$ with H_M, $\boldsymbol{P}_M$ operating in the superselection sectors $\mathscr{H}_M$. Now the vector valued distribution $\psi^+_{i_n}(x_n, t_n) \ldots \psi^+_{i_1}(x_1, t_1)\Omega$ with the $\{i_\nu\}$ varying over the different types of particles can be written in the form

$$\exp(-\mathrm{i}Ht_n)\exp(\mathrm{i}\boldsymbol{P}x_n)\psi^+_{i_n}(0)\exp[-\mathrm{i}H(t_{n-1}-t_n)]\exp[\mathrm{i}\boldsymbol{P}(x_{n-1}-x_n)]\psi^+_{i_{n-1}}(0)\ldots\Omega. \quad (13)$$

Each of the operators $H, \boldsymbol{P}$ in the exponents standing in between the creation operators can be replaced by a suitable H_M, $\boldsymbol{P}_M$ when applied to the vector standing on the right since each of the intermediate states carries a definite mass M. For these H_M, $\boldsymbol{P}_M$ we just derived analyticity properties, so we have the following theorem.

Theorem 4. $\psi^+_{i_n}(x_n, t_n) \ldots \psi^+_{i_1}(x_1, t_1)\Omega$ can be analytically continued with respect to the variables (x_n, t_n), $(x_{n-1} - x_n, t_{n-1} - t_n), \ldots$ into the domain τ^n, that is, (y_n, τ_n), $(y_{n-1} - y_n, \tau_{n-1} - \tau_n), \ldots \in \{(\tau, y); \tau > 0\}$, $i = 1, \ldots, n$.

The analyticity properties derived above allow for a proof of a Reeh–Schlieder property, well known from relativistic quantum field theory. Let $\mathcal{O} = \tilde{\mathcal{O}} \times I$ be a domain in $\mathbb{R}^4$, I an open time interval. Let $\mathscr{F}_{\mathcal{O}}$ denote the set of states generated by the application of creation operators localised in $\tilde{\mathcal{O}}$ at times $t \in I$ to the vacuum Ω:

$$\mathscr{F}_{\mathcal{O}} := \{\psi^+_n(f_n, t_n) \ldots \psi^+_1(f_1, t_1)\Omega\} \qquad \mathrm{supp}\, f_i \subset \tilde{\mathcal{O}} \qquad t_i \in I. \quad (14)$$

We want to show that this set is already dense in $\mathscr{H}$, that is in physical terms, almost all the relevant physics can already be generated in a finite space–time domain.

Theorem 5. The set $\mathscr{F}_{\mathcal{O}}$, defined by (14), is dense in $\mathscr{H}$.

Proof. We assume the contrary. That is, let $\varphi = \Sigma_M \varphi_M \neq 0$ be a vector orthogonal to $\mathscr{F}_{\mathcal{O}}$, hence φ_M orthogonal to $\mathscr{F}_{\mathcal{O}} \restriction \mathscr{H}_M$. With $\varphi \neq 0$ there exists a $\varphi_M \neq 0$. Obviously we can restrict ourselves to this sector $\mathscr{H}_M$. Let us form the scalar product

$$(\varphi_M | \psi^+_n(f_n, t_n) \ldots \psi^+_1(f_1, t_1)\Omega) \quad (15)$$

with the right vector an element of $\mathscr{H}_M$. We can choose the supports of $f_1, \ldots, f_n$ in such a way that for a_i varying over a small neighbourhood $\mathscr{U}_\varepsilon$ of $\mathbf{0} \in \mathbb{R}^3$, $f_i^{(a_i)}$ have still their supports in $\mathcal{O}(f_i^{(a_i)}(x) := f_i(x - a_i))$.

We have proved in theorem 4 that the function $F(a_n, t_n, \ldots, a_1 t_1) := (\varphi_M | \psi_n^+(f_n^{(a_n)}, t_n) \ldots \psi_1^+(f_1^{(a_1)}, t_1)\Omega)$ can be analytically continued to an open domain of $(\mathbb{C}^4)^n$ and that $F(a_n t_n, \ldots, a_1 t_1)$ are the boundary values for $\{\mathrm{Im}(z_i^0, \boldsymbol{z}_i)\} \to \mathbf{0}$. With our assumption made above, F vanishes on an open domain in $(\mathbb{R}^4)^n$ which is part of the boundary of T^n. Defining $G(\{z_i^0, \boldsymbol{z}_i\})$ by

$$G(\{z_i^0, \boldsymbol{z}_i\}) := \bar{F}(\{\overline{z_i^0}, \overline{\boldsymbol{z}_i}\}). \tag{16}$$

G is analytic in the tube $\bar{T}^n$ with a common real boundary set with F where $F \equiv G \equiv 0$. Hence we can infer that G is an analytic extension through a real open boundary set of the function F. Since this new function vanishes on a real open subset in the interior of the domain of definition it vanishes everywhere in the connected component of this set by standard reasoning of the theory of analytic functions of several complex variables (see e.g. Streater and Wightman 1964). By continuity also the boundary values are identically zero, that is

$$F(a_n t_n, \ldots, a_1 t_1) \equiv 0 \quad \text{everywhere in } (\mathbb{R}^4)^n. \tag{17}$$

By choosing the supports of the f_i appropriately and by freely shifting them we obtain, with the help of (17), that $(\varphi_M | \psi_n^+(x_n t_n) \ldots \psi_1^+(x_1 t_1)\Omega) \equiv 0$ in the sense of distributions for all M, n. Smearing now with arbitrary test functions we generate a dense set in $\mathscr{H}$ on the RHS hence $\varphi_M = 0$, that is $\varphi = 0$ which proves the theorem.

5. Space-like cluster properties and Galilei invariance

This investigation is almost independent of the results derived above. It is the counterpart of similar results in the relativistic case (see e.g. Jost and Hepp 1967). But perhaps somewhat surprisingly a spectrum condition will not be explicitly needed.

In (1) the transformation properties of the field operators under the full Galilei group were given. Smearing the fields with test functions from $\mathscr{S}$, that is, $\mathscr{C}^\infty$ functions which decrease strongly together with their derivatives, we can shift the action of the Galilei group to the test functions. While smearing with respect to space coordinates would be sufficient to guarantee the mere existence of the operators we have to smear with functions of $\mathscr{S}(\mathbb{R}^4)$ to infer differentiability properties with respect to the time. Denoting with $\mathscr{F}$ the dense subset of states generated by repeated application of the suitably smeared creation operators to the vacuum we see that this set is invariant under the action of the Galilei group and that repeated differentiation with respect to the parameters of the group $(a_0, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ is allowed. hence arbitrary powers of e.g. H, $\boldsymbol{P}$ leave $\mathscr{F}$ invaiant. This yields immediately the following lemma.

Lemma 2. The measure $(\varphi | \mathrm{d}E_\omega \, \mathrm{d}E_{\boldsymbol{k}} \psi)$ is strongly decreasing if $\varphi, \psi \in \mathscr{F}$.

Proof. With f a continuous bounded function on $\mathbb{R}^4$ we have

$$\left(\varphi \middle| \int f(\omega, \boldsymbol{k}) \, \mathrm{d}E_\omega \, \mathrm{d}E_{\boldsymbol{k}} \psi\right) \leq \|\varphi\| \cdot \|\psi\| \sup|f(\omega, \boldsymbol{k})|$$

and

$$\left(\varphi\middle|\int f(\omega,\boldsymbol{k})\omega^{n_0}\cdot\prod_i k_i^{n_i}\,\mathrm{d}E_\omega\,\mathrm{d}E_{\boldsymbol{k}}\psi\right)=\left(\varphi\middle|\int df\,\mathrm{d}E_\omega\,\mathrm{d}E_{\boldsymbol{k}}H^n\cdot\prod_i P_i^{n_i}\psi\right)$$

$$\leqslant\|\varphi\|\left\|H^{n_0}\cdot\prod_i P_i^{n_i}\psi\right\|\sup|f|. \tag{18}$$

Remark. The smearing with respect to time seems to be unavoidable:

$$U(a^0)\int\psi^+(x,t)f(x)\vartheta(t)\,\mathrm{d}x\,\mathrm{d}t\,U^{-1}(a^0)=\int\psi^+(x,t)f(x)\vartheta(t-a^0)\,\mathrm{d}x\,\mathrm{d}t \tag{19}$$

by which the differentiability with respect to t becomes obvious since $\vartheta\in\mathscr{S}(\mathbb{R}^1)$, (a sufficient continuity of the operator valued distribution assumed). On the other hand, the possibility of repeated application of H to $\psi_n^+(t_n,f_n)\ldots\psi_1^+(t_1,f_1)\Omega$ is not obvious, the time coordinate not being smeared.

However, specialising to the generators K_l of the Galilei boosts G_l, $l=1,2,3$, we see with the help of (1) that in this case a time smearing is not necessary since the time coordinates in the field operators are not affected. Hence, denoting with $\mathscr{F}'$ the dense set of states generated by repeated application of the creation operators to the vacuum, the time coordinates unsmeared, this set is left invariant by arbitrary powers of the K_l.

So let $f(\boldsymbol{P})$ be a bounded $\mathscr{C}^\infty$-function. With $\varphi,\psi\in\mathscr{F}'$ we have

$$(U(G_l)\varphi|f(\boldsymbol{P})U(G_l)\psi)=(\varphi|f(G_l^{-1}\cdot\boldsymbol{P})\psi)=(\varphi|f(\boldsymbol{P}-Mv_l\boldsymbol{e}_l)\psi) \tag{20}$$

$\boldsymbol{e}_l$ a unit vector in the l-direction, M the mass operator. But with $\mathscr{H}=\oplus\mathscr{H}_M$ we can study each sector $\mathscr{H}_M$ separately. The LHS is infinitely differentiable with respect to the components v_l of the velocity $\boldsymbol{v}$. On the RHS we have

$$(\partial_{v_l})^n|_{v_l=0}(\phi_M|f(\boldsymbol{P}-Mv_l\boldsymbol{e}_l)\psi_M)=\int(-M\partial_{k_l})^n f(\boldsymbol{k})(\phi_M|\,\mathrm{d}E_\omega\,\mathrm{d}E_{\boldsymbol{k}}\psi_M). \tag{21}$$

Remark. Note that as a result of the nonlinear v^2 term in the exponent of (1) additional terms show up in the higher derivatives which prevent the final result from being as smooth as e.g. in the related case of the Lorentz group. The same remark applies to a situation where f also depends on H.

Now we are ready to prove a strong cluster property with respect to the space coordinates.

Theorem 6. With $\phi,\psi\in\mathscr{F}'(\phi|U(\boldsymbol{a})\psi)$ is strongly decreasing for $|\boldsymbol{a}|\to\infty$.

Remark. Note that there is no vacuum polarisation because of the mass superselection rule, hence $(\phi_M|\Omega)=0$ for $M\neq 0$.

Proof.

$$(\phi|U(\boldsymbol{a})\psi)=\int\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{a})\,\mathrm{d}(\phi|E_{\boldsymbol{k}}\psi)$$

and

$$\int (-M\ \partial_{k_l})^n \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{a})\ \mathrm{d}(\phi|E_{\boldsymbol{k}}\psi)$$

is bounded by a constant $C_{n,\phi,\psi}$ as a result of (21) and (20). Hence we arrive at $|a_l|^n(\phi|U(a_l)\psi)$ which is bounded for $|a_l|\to\infty$ and every n.

Since it was not necessary to smear the various time coordinates occurring in the states ϕ, ψ we can in particular specialise to the subclass of usual n-body quantum mechanics. Furthermore, we have seen that the range of the potential enters nowhere openly in the theorem also Coulomb interactions are admitted. Hence we have the following corollary.

Corollary 3. Let ϕ, ψ be functions of $\mathscr{S}(\mathbb{R}^{3n})$. Then we have that $(\phi|\exp(\mathrm{i}Ht)\times \exp(\mathrm{i}\boldsymbol{Pa})\psi)$ is strongly clustering for $|\boldsymbol{a}|\to\infty$.

That is, while for $t\neq 0$ $\phi(t)$ need no longer be in $\mathscr{S}$ as a result of the interaction, its tails remain nevertheless asymptotically small. This is an extension of a result known to hold for the free time evolution.

Furthermore a short remark should be in order concerning a conjecture of Swieca (1967). In that paper it was argued that the (anti-)commutators of bounded operations should roughly decay in norm like the potential for the time coordinate held fixed, it should at least, somehow, be linked to the range of the potential. Our result, while incorporating the physically relevant expectation values of the field operators, is mathematically weaker than a decay in norm. But in any case there is no influence of the range of the potential on the cluster properties of the n-point functions. This is the analogue of a result known in relativistic quantum field theories but there the situation is a little bit nicer since the interaction is usually assumed to be local.

References

Emch G 1972 *Algebraic Methods in Statistical Mechanics and Field Theories* (New York: Interscience)
Hegerfeld G and Rysenaars S 1980 *Phys. Rev.* D **22** 377
Jost R and Hepp K 1967 *Helv. Phys. Acta.* **35** 34
Kato T 1966 *Perturbation Theories for Linear Operators* (New York: Springer)
Lévy-Leblond J M 1967 *Commun. Math. Phys.* **4** 157
—— 1971 in *Loebl: Group Theory* vol 2 (New York: Academic)
Reed M and Simon B 1975 *Fourier Analysis and Self Adjointness* (New York: Academic)
Streater R and Wightman A S 1964 *PCT, Spin and Statistics and All That* (New York: Benjamin)
Streit L 1969 *Nuovo Cimento* A **62** 673
Swieca J A 1967 *Commun. Math. Phys.* **4** 61
Wightman A S 1964 in *Cargese Lectures* ed M Levy (New York: Gordon and Breach)